Nullclines and phaseplanes

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In many cases, we will be able to reduce a system of differential equations to two independent variables in which case we have a planar system. Such systems have many advantages over higher-dimensional models. In particular, it is possible to qualtitatively sketch solutions without ever computing them. We will introduce three main concepts: (i) fixed points, (ii) stability, and (iii) nullclines. The latter are curves that enable one to break the plane into regions of different qualitative behavior. By phaseplane, I mean a sketch of the system in (x, y) coordinates. This is different from a sketch against time which is the way you often see data. Phaseplanes can provide much more useful info such as a global way of looking at your system. The pair of curves $(x(t), y(t)) = (\sin t, \cos t)$ when plotted against time is not obviously a circle; however, if you plot them *parametrically* in the plane, the geometry is clear. The same can be said for phaseplanes; they give "causal" information and provide a deeper understanding that time plots can ever produce.

We will assume the following system of equations must be solved:

$$x' = f(x, y) \tag{1}$$
$$y' = g(x, y) \tag{2}$$

where x' means the time-derivative of x. As an example to keep in mind while we do this, I will analyze the simple predator prey model

$$f(x, y) = x(2 - x - y) \quad g(x, y) = y(-1 + x)$$

In addition, I will suggest some XPP commands to make it easier to do this type of analysis.

STEP 1. Find the fixed points. That is solve, $f(x, y) = 0$ and $g(x, y) = 0$ for values of x, y. For the present model, it is easy to see that the following are the fixed points, $\{(0, 0), (2, 0), (1, 1).\}$ In general, the fixed points will not be so trivial to find, but that is what has to be done to start with.

STEP 2. Establish the stability of the fixed points. This is done by forming the matrix of partial derivatives evaluated at the fixed points:

$$A = \left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right],$$

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evaluating the components at the fixed points and examining the eigenvalues of the resulting matrix. (A theorem in ODEs says that linear stability implies nonlinear stability if none of the eigenvalues have zero real parts. Thus, the fixed point will be stable if all eigenvalues have negative real parts; is at least one eigenvalue has a positive real part, then the fixed point is unstable.) For 2×2 systems, it is easy to show that the eigenvalues of A have negative real parts if and only if, the determinant of A is positive and the trace of A is negative. (Recall that the determinant of $A = (a_{ij})$ is $\det A = a_{11}a_{22} - a_{12}a_{21}$ and the trace, $\text{Tr}A = a_{11} + a_{22}$.) Thus, it is trivial to determine stability. A few more obvious points: (a) if A is upper or lower triangular, the eigenvalues are read off the diagonals; (b) the methods in these two steps work in any dimension however the determination of the eigenvalues is more difficult in higher dimensions. I point out the following somewhat simple criterion for $n = 3$. Suppose the characteristic polynomial for A is $p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. Then the roots of p have negative real parts if and only if $a_2 > 0, a_0 > 0, a_1 a_2 > a_0$.

For the PP model, the matrix A is

$$A_{PP} = \begin{bmatrix} 2 - 2x - y & -x \\ y & -1 + x \end{bmatrix}$$

At the fixed point $(0,0)$, we see the eigenvalues are 2 and -1 so this fixed point is unstable. At the fixed point, $(2,0)$, we see that

$$A_{20} = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

which is upper triangular. The eigenvalues are -2 and 1 so it, too, is unstable. Finally, at the third fixed point we have:

$$A_{20} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

The trace of this is -1 and the determinant is 1, so this fixed point is stable.

STEP2b. If helpful, establish the nature of the fixed points. Let Δ be the determinant of A and T be the trace. The discriminant is $D = T^2 - 4\Delta$. There are three types of fixed points:

Nodes can be unstable or stable but have either two positive or two negative eigenvalues. These occur when $D > 0$.

Spirals or vortices can be unstable or stable but have two complex eigenvalues with positive or negative real parts and occur when $D < 0$. Solutions spiral into or out of these fixed points.

Saddles consist of one positive and one negative eigenvalue. They are unstable and occur when $\Delta < 0$. If a fixed point is a saddle then there are special solutions which merit mention. Consider the positive eigenvalue. Associated with this is the corresponding eigenvector. There exists a pair of solutions which leave the fixed point tangent to this eigenvector and these are called the *unstable manifold.* Similarly, there is a unique pair of solutions which come into the fixed point tangent to the eigenvector associated with the negative eigenvalue called the *stable manifold.* This is illustrated below:
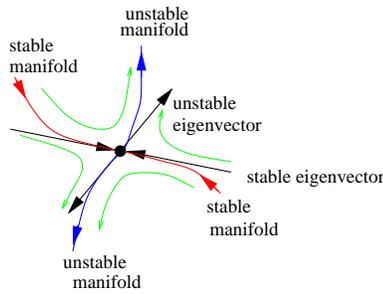
Here red (blue) trajectories are the stable (unstable) manifolds and the thick black arrows are the eigenvectors. Thin green lines are typical trajectories near the fixed points.

Based on this definition, in out PP example, there are two saddle-points, $(0,0)$ and $(2,0)$. We also have a stable vortex or spiral at the point $(1,1)$ since the discriminant is -3.

STEP 3 Draw the nullclines. Up to now, the phaseplane consists of a bunch of disjoint fixed points and perhaps some simple trajectories based on the local behavior. Thus, we want to glue this all together. Nullclines provide the skeleton that we can put the detailed meat onto. They provide a bulk picture of how things change at different points in the plane. The $x-$nullcline is the set of points in the plane where $f(x,y) = 0$. This naturally divides the plane into regions where $f(x,y) > 0$ and $f(x,y) < 0$. But recall that $x' = f(x,y)$ so that if $f(x,y) > 0$ that means x is increasing which in turn means it is moving rightward in the plane. Similarly, the $y-$nullcline is the set of points where $g(x,y) = 0$ and shows us where y is increasing, decreasing, and remaining constant. Intersections of nullclines are places where $f(x,y) = 0 = g(x,y)$ that is fixed points! The following table shows the directions of trajectories once the nullclines are

known:

What I usually do is find an easily identifiable point in the plane which is not on one of the nullclines. I evaluate (f,g) at this point and use this to draw a direction at that specific point. Then the rest of the arrows are easy to fill in using pure logic.

For our little PP example, the $x-$nullclines are the lines $x = 0$ (y-axis) and $y = 2-x$ while the $y-$nullclines are the lines $y = 0$ (x-axis) and $x = 1$. Let's complete our story and then compare it to the computer. Pick the point $(x,y) = (1/2, 1/2)$ and note that $(f,g) = (1/2, -1/4)$ so at this point the arrow points down and to the right. With this info and the nullclines I can complete the phaseplane:

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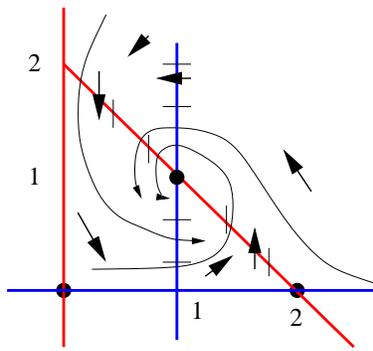
STEP 4 If you want - use the computer to do most of the work. Let's do the same thing using XPPAUT. You could also do this with Maple, Mathematica, MatLab, or even your own code. Here is the XPP file:

```
# pp.ode
# equations
x'=x*(2-x-y)
y'=y*(-1+x)
# set the screen dimensions
@ xlo=-.25,xhi=2.25,ylo=-.25,yhi=2.25
# set nullcline mesh
@ nmesh=200
# set plotting variables
@ xp=x,yp=y
# set the background
@ background=white
done
```

Most of the file is comments. It is set up to display solutions in the phaseplane. If you click `Nullcline; New` you will see the nullclines drawn in green and red. Next click on `Dir.Fld/flow;Scaled` and accept the default grid of 10 to see a bunch of arrows appear. Now click on `Initialconds;mIce` and click in different places in the plane to see trajectories. Click `Esc` when you are done. In MSWin, you can probably grab the screen to make a printer dump. I usually do the following: `Kinescope;Capture` which does a screen grab. Then `Kinescope;Save` in the GIF format to save the frame. You can view this with Netscape or Explorer. I use XV and convert it to postscipt. Here is what I get:

Y vs X
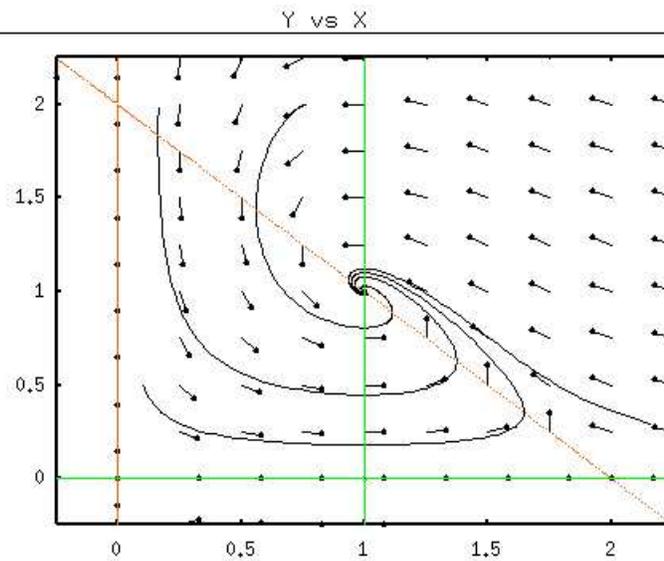
As you can see, our crude picture was pretty close.

Here is another example which we can qualitatively analyze:

$$x' = 4x(1 - x^2) - y \qquad y' = x - y.$$

STEP 1. The fixed points satisfy, $y = x$ so this means $4x(1 - x^2) = x$ or that $x = 0 = y$ or $4(1 - x^2) = 1$ implying $x = y = \pm\sqrt{3}/2$ so there are 3 fixed points.

STEP 2. The linearized matrix is

$$A = \begin{bmatrix} 4 - 12x^2 & -1 \\ 1 & -1 \end{bmatrix}$$

which for the $(0, 0)$ fixed point yields

$$A_{0,0} = \begin{bmatrix} 4 & -1 \\ 1 & -1 \end{bmatrix}$$

Since the determinant is -3, this is a saddle point. For the other two points,

$$A_{*,*} = \begin{bmatrix} -5 & -1 \\ 1 & -1 \end{bmatrix}$$

which has a determinant of 6 a trace of -6 and a discriminant of 12 so it is a stable node.

STEP 3. The x−nullcline is the curve, $y = 4x(1 - x^2)$ which is a cubic and the y−nullcline is the line $y = x$. Pick a point, say, $x = 0, y = 1$ and find that $(f, g) = (-1, -1)$ which points down and to the left. The rest is easy to fill in:

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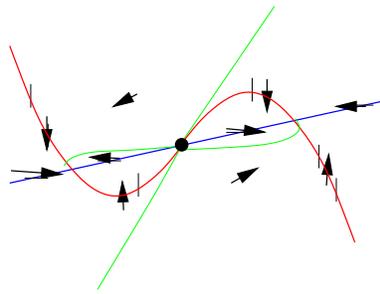
I have attempted to sketch the stable and unstable manifolds for the system in green lines. The stable manifold breaks the plane into two parts to the left of it everything goes to the lower fixed point and to the right, all solutions go to the upper point.

STEP 4. Here is what the computer yields:
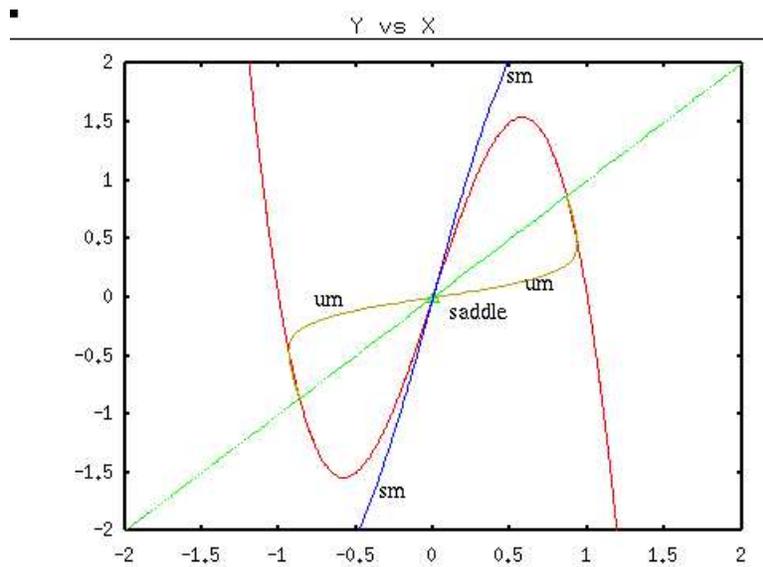
Note that XPP allows you to compute the stable/unstable manifolds of saddle points. Click on `Sing.Pts.; Go` to compute stability of fixed points and when appropriate, the invariant sets. The blue (turquoise in XPP) curves are the stable manifolds (sm) and the brown (yellow in XPP) are the unstable manifolds (um).

HOMEWORK

1. Fill in these phaseplanes as well as you can given the nullcline information and the arrows. Point out where possible which points are stable and unstable.

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A   B

C   D
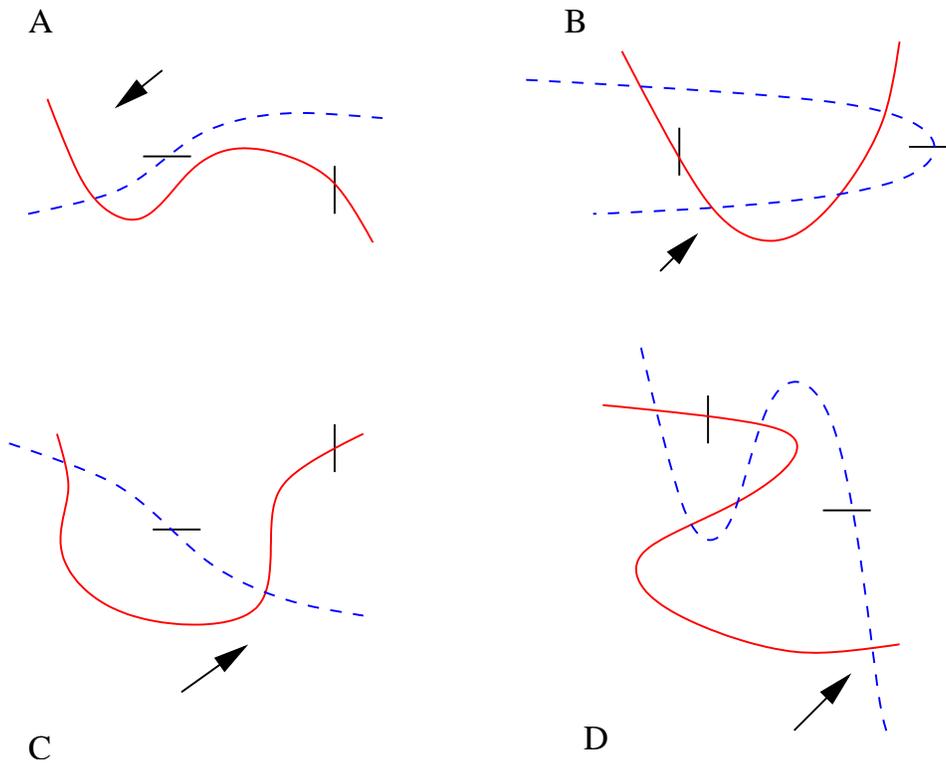
2. Many biological systems are what are called "activator-inhibitors" and their linearized matrix has the form:

$$A = \begin{bmatrix} \pm a & -b \\ -c & -d \end{bmatrix}$$

where a, b, c, d are all positive.

(a) Suppose we take the negative sign for a. Show that the fixed point is stable in this case. Sketch the nullclines for the system $x' = -ax - by$ and $y' = cx - dy$.

(b) Suppose that we take the positive sign for a. Then both the $x-$ and $y-$nullclines have positive slopes. Show that is the $x-$nullcline is steeper than the $y-$nullcline, the fixed point is a saddle. Otherwise show that the position of the nullclines cannot tell you definitive stability information.

3. Analyze the fixed points and their stability and draw the phaseplane and nullclines for the following system which depends on a parameter, a:

$$x' = x(x(1-x) - y) \qquad y' = y(x - a)$$

Note that $a > 0$ and that this is a predator prey model. Try $a = .75, a = .4$ and make sure you integrate for a long time for the latter (say 100 time units).

4. Ditto for this system

$$S' = -\beta SI + \gamma(1 - S - I) \qquad I' = \beta IS - \delta I$$
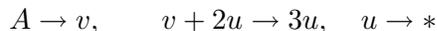
where β, γ, δ are all positive parameters.

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5. Study the fixed points and stability for the biochemical model:

$$x' = a - xy + \frac{x}{x+1} \qquad y' = yx - y$$

Also for this model, write down a set of possible chemical reactions (allowing for fast enzymatic reactions if needed) that will lead to this model.

6. Derive equations for the simple chemical model

$$A \to v, \qquad v + 2u \to 3u, \qquad u \to *$$

and assume all rates are 1. You should get two ODEs for (u, v). Find all the fixed points as a function of the parameter, A and study their stability. What happens if the fixed point is unstable? Sketch the phaseplane in this case and numerically compute some orbits for $A = .95$ for example.

7. Last but not least, let us return to the problem we defined in class for the biochemical oscillator

$$x' = y - \mu x + \frac{cx^2}{d + x^2} \qquad y' = k(1 - bxy)$$

Choosing the parameters, $\mu = 1, d = 6.5, k = 0.01$ look at the phase plane and numerically compute some solutions for values of c between 4.5 and 4.8. Can you find periodic solutions or multiple stable fixed points? You may have to integrate for a long time. Here is an XPP file

```
x'=y-mu*x+c*x^2/(d+x^2)
y'=(1-b*x*y)*k
par mu=1,c=4.6,d=6.5,k=.01,b=2.7
@ xp=x,yp=y,xlo=-.1,ylo=-.1,xhi=5,yhi=2
# tell XPP to use an adaptive step integrator - better accuracy!
@ meth=qualrk,tol=1e-6,total=400,dt=.5
done
```