Math 241, Exam 2. 10/24/12. **Name:** ______________

- Read problems carefully. Show all work. No notes, calculator, or text.
- The exam is roughly 15 percent of the total grade. There are 100 points total.
- Write your full name in the upper right corner of page 1.
- Do problem 1 on p. 1, problem 2 on p. 2,...(or at least present your solutions in numerical order).
- Circle or otherwise clearly identify your final answer.

1. **(15 points):** Suppose that $z = f(x, y)$, where

$$x = g(s,t), \quad y = h(s,t), \quad g(1,2) = 3, \quad g_s(1,2) = -1,$$

$$h(1,2) = 6, \quad h_s(1,2) = -5, \quad f_x(3,6) = 7, \quad f_y(3,6) = 8.$$

Use the chain rule to find $\partial z/\partial s$.

Solution: We have

$$\frac{\partial z}{\partial s} = f_x(x,y)g_s(s,t)+f_y(x,y)h_s(s,t) = f_x(3,6)g_s(1,2)+f_y(3,6)h_s(1,2) = (7)(-1)+(8)(-5) = -47.$$

2. **(15 points):** Consider the surface with equation $\sin(xyz) = x + 2y + 3z$. Find an equation for the plane tangent to the surface at the point $(2, -1, 0)$.

Write your answer in the form $ax + by + cz = d$.

Solution: We have $F(x, y, z) = \sin(xyz) - x - 2y - 3z$; it follows that

$$\nabla F(x,y,z) = \langle yz\cos(xyz) - 1, xz\cos(xyz) - 2, xy\cos(xyz) - 3\rangle.$$

Hence, we have

$$\nabla F(2,-1,0) = \langle -1, -2, -5\rangle.$$

The tangent plane has equation

$$-1(x-2) - 2(y+1) - 5(z-0) = 0 \iff -x - 2y - 5z = 0 \iff x + 2y + 5z = 0.$$

3. **(25 points):** Let $f(x, y) = x^2y - 2y^{3/2}$.

   (a) **(9 points):** Find the maximum rate of change of $f(x, y)$ at the point $(1, 4)$.
   In which direction does it occur? (Specify a vector; it need not be a unit vector.)

   Solution: We compute $\nabla f(x, y) = \langle 2xy, x^2 - 3y^{1/2}\rangle$. It follows that the direction of maximum rate of change at $(1, 4)$ is $\nabla f(1, 4) = \langle 8, -5\rangle$, while the maximum rate of change at $(1, 4)$ is $\|\nabla f(1, 4)\| = \|\langle 8, -5\rangle\| = \sqrt{89}$.

(b) **(8 points):** Suppose that the $D_{\vec{v}}f(1,4)$ is half of its maximum value (the value from part (a)). What is the angle $0 \leq \theta \leq \pi$ (in radians) that $\vec{v}$ makes with the vector that points in the direction of maximum increase (the direction from (a))?

<u>**Solution**</u>**:** We have $D_{\vec{v}}f(1,4) = \nabla f(1,4) \cdot \vec{v} = \|\nabla f(1,4)\| \cos\theta = (1/2)\|\nabla f(1,4)\|$ if and only if $\cos\theta = 1/2$, which holds if and only if $\theta = \pi/3$.

(c) **(8 points):** Find a unit vector $\vec{u} = \langle u_1, u_2 \rangle$ for which $D_{\vec{u}}f(1,4) = 0$.
(You may use your work from (a), if you choose.)

<u>**Solution**</u>**:** We have

$$D_{\vec{u}}f(1,4) = \nabla f(1,4) \cdot \langle u_1, u_2 \rangle = \langle 8, -5 \rangle \cdot \langle u_1, u_2 \rangle = 8u_1 - 5u_2 = 0 \iff u_2 = (8/5)u_1.$$

Since $\|\vec{u}\| = 1$, we also have

$$1 = u_1^2 + u_2^2 = u_1^2 + 64u_1^2/25 \iff 89u_1^2 = 25 \iff u_1 = \frac{5}{\sqrt{89}}, \quad u_2 = \frac{8}{\sqrt{89}}.$$

Hence, a unit vector with the desired property is $\vec{u} = (1/\sqrt{89})\langle 5, 8 \rangle$.

4. **(15 points):** The function $f(x,y) = 3xy - x^2y - xy^2$ has four crititical points. Three of them are $(0,0)$, $(3,0)$, and $(0,3)$.

(a) **(7 points):** Find the fourth critical point.

<u>**Solution**</u>**:** We compute $f_x = 3y - 2xy - y^2$ and $f_y = 3x - x^2 - 2xy$. We solve $f_x = 0$, $f_y = 0$ by subtracting the first two equations, we get $0 = 3y - 3x - y^2 + x^2 = (x-y)(x+y) - 3(x-y) = (x-y)(x+y-3)$ which holds if and only if $x = y$ or $x + y = 3$. If $x = y$, then we have $f_x = 0 = 3y - 2xy - y^2 = 3x - 2x^2 - x^2 = 3x - 3x^2 = 3x(1-x)$. In this case, we get $x = y = 0$ or $x = y = 1$. Hence, the fourth point is $(1,1)$. We also get the point $(0,0)$. If $y = 3 - x$, then we have $f_y = 0 = 3x - x^2 - 2x(3-x) = 3x - x^2 - 6x + 2x^2 = x^2 - 3x = x(x-3)$. In this case, we get $x = 0$, $y = 3$ and $x = 3$, $y = 0$, accounting for all critical points.

(b) **(8 points):** Identify the critical point $(0,0)$ and the critical point from (a) as local maxima, local minima, saddle points, or neither.

<u>**Solution**</u>**:** We compute $f_{xx} = -2y$, $f_{yy} = -2x$, and $f_{xy} = 3 - 2x - 2y$. We have

$\mathbf{(0,0)}$: $D(0,0) = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = 9 < 0$; hence, $(0,0)$ is a saddle point.

$\mathbf{(1,1)}$: $D(1,1) = \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 3 > 0$; since $f_{xx}(1,1) = -2 < 0$, the point $(1,1)$ is a local maximum.

5. **(10 points):** Compute the double integral $\iint_D x\cos(xy)\,dA$, where D is the rectangle bounded by $0 \le x \le \pi/2$ and $0 \le y \le 2$.
(Choose carefully whether to use $dA = \,dx\,dy$ or $dA = \,dy\,dx$.

<u>Solution</u>: We compute

$$\iint_D x\cos(xy)\,dA = \int_0^{\pi/2}\int_0^2 x\cos(xy)\,dy\,dx = \int_0^{\pi/2} x\left(\frac{1}{x}\sin(xy)\right)\Bigg]_{y=0}^{2}\,dx$$
$$= \int_0^{\pi/2}\sin(2x)\,dx = -\frac{1}{2}\cdot\cos(2x)\Bigg]_{x=0}^{\pi/2} = -\frac{1}{2}\cdot(-1-1) = 1.$$

6. **(20 points):** Use Lagrange Multipliers to find the minimum value of

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to the constraint $g(x,y,z) = y^2 - xz - 9 = 0$.

<u>Solution</u>: We compute $\nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle$ and $\nabla g(x,y,z) = \langle -z, 2y, -x\rangle$. Therefore, we solve the system

$$2x = -\lambda z, \quad y = \lambda y, \quad 2z = -\lambda x, \quad y^2 - xz = 9.$$

The second equation gives $\lambda = 1$ or $y = 0$.

$\boldsymbol{y \neq 0}$: We have $\lambda = 1$. Equations (1) and (3) become $2x = -z$ and $2z = -x$. We solve for x in the first and substitute in the second to obtain $2z = z/2$, which gives $z = 0$; hence, we also have $x = 0$. We substitute in the constraint equation to get $y^2 = 9$. It follows that $(0, \pm 3, 0)$ are critical points. The value of f at these points is 9.

$\boldsymbol{y = 0}$: Neither x nor $z = 0$. (If either were zero, then the constraint equation reads $0 = 9$, a contradiction.). We may now divide by x, $z \neq 0$ in equations (1) and (3) to obtain $2x/z = 2z/x$, from which we deduce that $x^2 = z^2$. Therefore, we have $x = \pm z$. We substitute $x = z$ in the constraint equation to get $-x^2 = 9$, which has no solution. With $x = -z$ in the constraint equation, we get $x^2 = 9$, so $x = \pm 3$. It follows that $(3, 0, -3)$ and $(-3, 0, 3)$ are critical points. The value of f at these points is 18.

We conclude that the minimum value of f subject to the constraint $g = 0$ is $f(0, \pm 3, 0) = 9$.